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Entropy of Gibbons–Maeda Dilaton Black Hole due to Arbitrary Spin Fields

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Using the membrane model which is based on brick-wall model, we calculated the free energy and entropy of Gibbons–Maeda dilation black hole due to arbitrary spin fields. The result shows that the entropy of a scalar field and the entropy of a fermion field have similar formulas. There is only a coefficient difference between them. Furthermore, both entropies depend on the degeneracy of the field.

KEY WORDS: black hole entropy; brick-wall model; membrane model; spin.

1. INTRODUCTION

In theoretical physics, the thermodynamics of black holes remains an enigma, it turns out to be a junction of general relativity, quantum mechanics, and statistical physics.

Since Bekenstein and Hawking proposed, in 1970s, that the black hole entropy is proportional to the area of the event horizon (Bekenstein, 1972, 1973, 1974; Hawking, 1975; Kallosh, 1993), much efforts are devoted to study the statistical origin of the black hole entropy. One such effort is the widely used brick wall model proposed by 't Hooft (1985). By using this model, 't Hooft investigated the statistical properties of a free scalar field in the Schwarzschild black hole background and obtained an expression for entropy in terms of the area of the event horizon which verifies the proportional relationship between them. Furthermore, when the cutoff parameter satisfies a certain condition, the entropy can be written as $S = A_h/4$, while for the case when the cutoff parameter tends to zero, the entropy would

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be divergent, which was explained as due to the infinite density of states at the vicinity of the horizon. Another different but actually equivalent approach (Callan and Wilczek, 1994; Kabat and Strassler, 1994) is adopted by (Bombelli *et al.* (1986) and Srednicki (1993). Solodukhin used Gibbons–Hawking Euclidean path integral approach (Gibbons and Hawking, 1977) to study the quantum corrections to the entropy of a Schwarzschild black hole (Solodukhin, 1995a,b) starting with the one-loop effective action of scalar matter. In quantum mechanics, geometric entropy satisfies the following assumptions: If particles are scalar bosons obeying Bose–Einstein statistics, the entropy obtained is conventionally called the boson entropy; if the quantum-mechanical geometric entropy is calculated by counting the fermions particle states, the corresponding entropy is called the fermion entropy.

Since the mid of 1990s, such problems have aroused much interest among many researchers (Brown, 1995; Carlip, 1995; Carlip and Teitelbion, 1995; Cretic and Youm, 1996; Cognola and Lecca, 1998; de Alwis and Ohta, 1995; Demers *et al.*, 1995; Fila *et al.*, 1994; Gao and Shen, 2001; Gao and Shen, 2002a,b; Ghosh and Mitra, 1994, 1995; Gubser *et al.*, 1996; Hawking *et al.*, 1995; Ichinose and Satoh, 1995; Jacobson *et al.*, 1995; Kabat *et al.*, 1995; Kim, 1997; Larssen and Wilczek, 1995, 1996; Lee and Kim, 1996a,b; Lee *et al.*, 1996b; Mann and Solodukhin, 1996; Pinto-Neto and Soares, 1995; Russo, 1995; Shen, 2000a,b,c; Shen, 2002; Shen and Chen, 1998a,b; Shen and Chen, 1999a,b,c; Shen and Chen, 2000, 2001; Shen and Cheng, 2001; Shen and Gao, 2002a,b; Shen *et al.*, 1997; Solodukhin, 1995a,b,c; Solodukhin, 1996; Srednicki, 1993; Susskind and Uglum, 1994; Teitelbion, 1995; Zhou *et al.*, 1995). But up to now, the method mainly used by people is brick wall model; furthermore, to get a result proportional to the area, we must use the small-mass approximation and neglect the part nonproportional to area. Considering the divergence of the wave function near the event horizon and the introduction of cutoff, why not assume that the free energy of black hole only comes from a layer in the infinity of event horizon? Such a physical picture is very obvious.

Membrane model (Gao and Liu, 2000; Li and Zhao, 2000) assumes that the region in which wave function is not zero is $r_H + \epsilon \leq r \leq r_H + \epsilon + \delta$. That is, the integral interval of r is only in a membrane.

In this paper, by using a membrane model, we obtained the free energy and entropy of gravitational field (spin $s = 2$), electromagnetic field (spin $s = 1$) and neutrino field (spin $s = \frac{1}{2}$). The formulas of entropy are given. We found that the entropy of scalar field and fermions field have similar formulas. There is only a coefficient difference between them. This is similar to the former (Cognolla and Lecca, 1998; de Alwis and Ohta, 1995; Larsen and Wilczek, 1995; Shen, 2000a; Shen and Chen, 1999b,c; Zhou *et al.*, 1995). But both entropies depend on the degeneracy of the fields.

2. FIELD EQUATION

The metric of Gibbons–Maeda dilaton black hole writes (Gibbons and Moeda, 1988; Solodukhin, 1995b)

$$ds^2 = \frac{(r - r_+)(r - r_-)}{R^2}\, dt^2 - \frac{R^2}{(r - r_+)(r - r_-)}\, dr^2 - R^2(d\theta^2 + \sin^2\theta)\, d\varphi^2, \tag{1}$$

where

$$r_\pm = M \pm \sqrt{M^2 + D^2 - P^2 - Q^2}, \tag{2}$$

$$D = \frac{P^2 - Q^2}{2M}, \tag{3}$$

$$R^2 = r^2 - D^2. \tag{4}$$

The parameters M, Q, and P are the mass, the electronic charge, and the magnetic charge of the hole, respectively.

Choose the null tetrad as follows:

$$l^\mu = \left(\frac{R^2}{(r - r_+)(r - r_-)}, 1, 0, 0\right), \tag{5}$$

$$n^\mu = \frac{1}{2}\left(1, -\frac{(r - r_+)(r - r_-)}{R^2}, 0, 0\right), \tag{6}$$

$$m^\mu = \frac{1}{\sqrt{2}R}\left(0, 0, 1, \frac{i}{\sin\theta}\right), \tag{7}$$

$$\bar{m}^\mu = \frac{1}{\sqrt{2}R}\left(0, 0, 1, -\frac{i}{\sin\theta}\right). \tag{8}$$

The corresponding covariant null tetrad is

$$l_\mu = \frac{1}{2}\left(1, -\frac{R^2}{(r - r_+)(r - r_-)}, 0, 0\right), \tag{9}$$

$$n_\mu = \frac{1}{2}\left(\frac{(r - r_+)(r - r_-)}{R^2}, 1, 0, 0\right), \tag{10}$$

$$m_\mu = -\frac{R}{\sqrt{2}}(0, 0, 1, i\sin\theta), \tag{11}$$

$$\bar{m}_\mu = -\frac{R}{\sqrt{2}}(0, 0.1, -i\sin\theta). \tag{12}$$

The above null tetrad satisfies null vector conditions

$$l_\mu l^\mu = n_\mu n^\mu = m_\mu m^\mu = \bar{m}_\mu \bar{m}^\mu = 0; \tag{13}$$

pseudo-orthogonality conditions

$$l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1, \tag{14}$$

$$l_\mu m^\mu = l_\mu \bar{m}_\mu = n_\mu m^\mu = n_\mu \bar{m}^\mu = 0; \tag{15}$$

and metric conditions

$$g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu. \tag{16}$$

The nonvanishing spin coefficients are (Garfinkle *et al.*, 1991)

$$\rho = -\frac{r}{r^2 - D^2}, \tag{17}$$

$$\alpha = -\beta = -\frac{1}{2\sqrt{2}\sqrt{r^2 - D^2}} ctg\theta, \tag{18}$$

$$\mu = -\frac{1}{2}\frac{r(r - r_+)(r - r_-)}{(r^2 - D^2)^2}, \tag{19}$$

$$\gamma = \frac{1}{4}\left[\frac{2r - (r_+ - r_-)}{r^2 - D^2} - \frac{2r(r - r_+)(r - r_-)}{(r^2 - D^2)^2}\right]. \tag{20}$$

Only one of Weyl tensors is not zero, i.e.,

$$\Psi_2 = -\frac{r}{2}\frac{2r - (r_+ - r_-)}{(r^2 - D^2)^2} + \frac{r^2(r - r_+)(r - r_-)}{(r^2 - D^2)^3} + \frac{D^2}{2}\frac{(r - r_+)(r - r_-)}{(r^2 - D^2)^3}. \tag{21}$$

Equations (17–21) show that the GHS (Garfinkle–Horowitz–Strominger) metric is of Petrov-type D. Using the result of Teukolsky (Newman and Penrose, 1962; Teukolsky, 1973), the field equation of spins $= \frac{1}{2}$, 1, and 2 for the source free case can be combined into

$$\begin{aligned}
&[D - (2s + 1)\rho][\Delta - 2s\gamma + \mu]\Phi_{+s} \\
&- \{[\delta + (2s - 2)\alpha][\bar{\delta} - 2s\alpha] + (2s - 1)(s - 1)\Psi_2\}\Psi_{+s} = 0, \\
&[\Delta + (2s - 2)\gamma + (2s + 1)\mu][D - \rho]\Phi_{-s} \\
&- \{[\delta + (2\bar{s} - 2)\alpha][\delta - 2s\alpha] + [2s - 1](s - 1)\Psi_2\}\Phi_{-s} = 0,
\end{aligned} \tag{22}$$

where

$$D \equiv l^\mu \partial_\mu = \frac{R^2}{(r - r_+)(r - r_-)}\partial_t + \partial_r, \tag{23}$$

$$\Delta \equiv n^{\mu}\partial_{\mu} = \frac{1}{2}\partial_t - \frac{1}{2}\frac{(r-r_+)(r-r_-)}{R^2}\partial_r, \tag{24}$$

$$\delta \equiv m^{\mu}\partial_{\mu} = \frac{1}{\sqrt{2}R}\left(\partial_{\theta} + \frac{i}{\sin\theta}\partial_{\varphi}\right), \tag{25}$$

$$\bar{\delta} \equiv \bar{m}^{\mu}\partial_{\mu} = \frac{1}{\sqrt{2}R}\left(\partial_{\theta} - \frac{i}{\sin\theta}\partial_{\varphi}\right). \tag{26}$$

The first one of Eqs. (22) is for spin states $p = s$ and the other one for $p = -s$ respectively. Make transformations below (Newman and Penrose, 1962)

$$\Phi_{+s}, \Phi_{-s} = r^{p-s}\, {}_pR_{lE}(r)\, {}_pY_l^m(\theta, \varphi)\, e^{-iEu}. \tag{27}$$

Put Eqs. (17–21), (23–27) into Eqs. (22), we obtain the radial equation

$$\begin{aligned}
&(r-r_+)(r-r_-)\partial_r^2\, {}_pR_{lE}(r) + (p+1)(2r-r_+-r_-)\partial_r\, {}_pR_{lE}(r) \\
&+ p\frac{r(r-r_+)(r-r_-)}{r^2-D^2}\partial_r\, {}_pR_{lE}(r) \\
&+ \left[E^2\frac{(r^2-D^2)^2}{(r-r_+)(r-r_-)} + A(r) + iEB(r) - \lambda^2\right]{}_pR_{lE} = 0,
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
A(r) = &-\frac{2pD^2}{r^2-D^2} + \frac{2p(4p-5)r^2}{r^2-D^2} - \frac{(4p^2+3)(r_++r_-)r}{r^2-D^2} \\
&- \frac{(2p+1)r_+r_-}{r^2-D^2} + (4p^2+3p+7)\frac{r^2(r-r_+)(r-r_-)}{(r^2-D^2)^2} \\
&+ (2p^2-3p+1)\frac{D^2(r-r_+)(r-r_-)}{(r^2-D^2)^2},
\end{aligned} \tag{29}$$

$$B(r) = 4pr - p(r^2-D^2)\frac{2r-(r_+-r_-)}{(r-r_+)(r-r_-)}, \tag{30}$$

and the angular equation

$$\begin{aligned}
&\left[\frac{1}{\sin\theta}\partial_{\theta}(\sin\theta\partial\theta) + \frac{1}{\sin^2\theta}\partial_{\varphi}^2 + \frac{2ip\cos\theta}{\sin^2\theta}\partial_{\varphi}\right]{}_pY_l^m(\theta,\varphi) \\
&-[p^2ctg^2\theta + p - \lambda^2]_pY_l^m(\theta,\varphi) = 0.
\end{aligned} \tag{31}$$

Equation (31) shows that ${}_pY_l^m$ is the spin-weighed spherical harmonic (Goldberg *et al.*, 1967; Teukolsky and Press, 1974) and the separation constant λ satisfies

$$\lambda = \sqrt{(l-p)(l+p+1)}, \tag{32}$$

where

$$l \geq |p|, \qquad -l \geq m \geq l. \tag{33}$$

3. FREE ENERGY AND ENTROPY

In this section we will calculate the black hole entropy via membrane model. As a simplest model, membrane model assumes that in the vicinity of event horizon there is a layer of radiation whose thickness is δ and whose distance to the event horizon is ε. The entropy of black hole is identified with that of the membrane. So the boundary condition of the wave function reads

$$\Phi(r) = 0, \qquad \text{when } r \leq r_+ + \varepsilon, \tag{34}$$

$$\Phi(r) = 0, \qquad \text{when, } r \geq r_+ + \varepsilon + \delta, \tag{35}$$

where $\varepsilon \ll r_+$, $\delta \ll r_+$, r_+ is the event horizon of black hole.

Let

$$_pR_{lE}(r) = e^{iZ}, \tag{36}$$

and using WKB approximation, we obtain

$$K^2 = (\partial_r Z)^2 = \frac{(r^2 - D^2)^2}{(r - r_+)(r - r_r)} E^2 + \frac{1}{(r - r_+)(r - r_-)} A(r) - \frac{(l - p)(l + P + 1)}{(r - r_+)(r - r_-)}, \tag{37}$$

where K is the radial numbers of wave.

The constraint of semiclassical quantum condition imposed on K is given by

$$n\pi = \int_{r_+ + \varepsilon}^{r_+ + \varepsilon + \delta} K\, dr, \tag{38}$$

where n is a nonnegative integer. As same as the brick wall model, energy E is positive and wave number K is real.

According to the ensemble theory, the free energy is given by

$$\beta F = \mp \sum \ln(1 \pm e^{-\beta\omega}), \tag{39}$$

where β is the inverse of Hawking temperature, i.e.,

$$T_{\mathrm{H}} = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \frac{r_+ - r_-}{r_+^2 - D^2}. \tag{40}$$

Looking the states of energy as continuous and transform summation into integration, we obtain

$$\sum \to \int_0^{\infty} dEg\,(E), \tag{41}$$

where $g(E)$ is the density of states, i.e.

$$g(E) = \frac{d\Gamma(E)}{dE}, \tag{42}$$

$\Gamma(E)$ is the number of the microscopic states, i.e.

$$\Gamma(E) = \sum_p \sum_l (2l+1)\,n. \tag{43}$$

Transforming the summation of l into integration and requiring $K \geq 0$, then we obtain

$$\begin{aligned}
\Gamma(E) &= \sum_p \int (2l+1)\,dl \frac{1}{\pi} \int K dr \\
&= \frac{1}{\pi} \sum_p \int_{r_+ + \varepsilon}^{r_+ + \varepsilon + \delta} dr \int_{|p|}^{l_{\max}} dl(2l+1)[(r-r_+)(r-r_-)]^{-\frac{1}{2}} \\
&\quad \left[(r^2-D^2)^2E^2 + A(r) - (l-p)(l+p+1)\right]^{\frac{1}{2}} \\
&= \frac{2}{3\pi} \sum_P \int_{r+h+\varepsilon}^{r_+ + \varepsilon + \delta} dr[(r-r_+)(r-r_-)]^{-\frac{1}{2}} \\
&\quad [(r^2-D^2)^2E^2 + A(r) - (|p|-p)]^{\frac{3}{2}}.
\end{aligned} \tag{44}$$

The free energy can be written as

$$F = -\frac{2}{3\pi}\frac{1}{\beta} \sum_P \int_0^{\infty} \frac{E^3 dE}{e^{\beta E} \mp 1} \int_{\tau+h+\varepsilon}^{r_+ + \varepsilon + \delta} dr \frac{(r^2-D^2)^3}{(r-r_+)^2(r-r_-)^2}, \tag{45}$$

$$F_{\text{bosons}} = -\frac{4\omega\pi^3}{90\beta^4} \int_{r_+ + \varepsilon}^{r_+ + \varepsilon + \delta} dr \frac{(r^2-D^2)^3}{(r-r_+)^2(r-r_-)^2}, \tag{46}$$

$$F_{\text{fermions}} = -\frac{7}{8}\frac{4\omega\pi^3}{90\beta^4} \int_{r_+ + \varepsilon}^{r_+ + \varepsilon + \delta} dr \frac{(r^2-D^2)^3}{(r-r_+)^2(r-r_-)^2}, \tag{47}$$

where ω is the degeneracy due to spin. For the gravitational and electromagnetic fields we have $\omega = 2$; for the neutrino and scalar fields we have $\omega = 1$.

Considering the relation between entropy and free energy below

$$S = \beta^2 \frac{\partial F}{\partial \beta}, \tag{48}$$

we obtain (when $r_+ \neq r_-$)

$$F_{\text{bosons}} = \frac{\omega}{360} \cdot \frac{r_+ - r_-}{\varepsilon} \cdot \frac{\delta}{\varepsilon + \delta}, \tag{49}$$

$$S_{\text{fermions}} = \frac{7}{8}\frac{\omega}{360} \cdot \frac{r_+ - r_-}{\varepsilon} \cdot \frac{\delta}{\varepsilon + \delta}. \tag{50}$$

Put

$$\delta_0^2 = \frac{2\varepsilon_0^2}{15}, \tag{51}$$

where

$$\delta_0^2 = \int_{r_+}^{r_+ + \varepsilon} \sqrt{\frac{r^2 - D^2}{(r - r_+)(r - r_-)}}\, dr \approx 2\sqrt{\varepsilon \frac{r_+^2 - D^2}{r_+ - r_-}} \tag{52}$$

is the proper distance from r_+ to $r_+ + \varepsilon$. By Eqs. (50–53), we obtain

$$S_{\text{bosons}} = \omega \cdot \frac{A_h}{48\pi \epsilon_0^2} \frac{\delta}{\varepsilon + \delta}, \tag{53}$$

$$S_{\text{ fermions}} = \frac{7}{8}\omega \cdot \frac{A_z}{48\pi \varepsilon_0^2 \varepsilon} \frac{\delta}{\varepsilon + \delta}, \tag{54}$$

where

$$A_h = 4\pi (r_+^2 - D^2) \tag{55}$$

is the area of the event horizon.

For the extreme black hole $(r_+ - r_-)$, the area of the event horizon is zero, however, the entropy is not zero, i.e.,

$$S_{\text{bosons}}^{\text{ext}} = \omega \left(\frac{r_+^2 - D^2}{\beta \varepsilon} \right)^3 \ln \left(1 + \frac{\delta}{\varepsilon} \right), \tag{56}$$

$$S_{\text{fermions}}^{\text{ext}} = \frac{7}{8}\omega \left(\frac{r_+^2 - D^2}{\beta \varepsilon} \right)^3 \ln \left(1 + \frac{\delta}{\varepsilon} \right). \tag{57}$$

The area A_h of the event horizon is zero while the entropy S_{ext} is not zero. In particular, when $P = 0$, $M^2 = \frac{Q^2}{2}$.

Equations (54–55) and (57–58) show that Gibbons–Maeda dilaton black hole entropies due to scalar field ($s = 0, 1, 2$) and fermions field ($s = \frac{1}{2}$) have similar formulas. There is only a coefficient difference between them. They both depend on the degeneracy of the fields.

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